

Real Rank Two Geometry

Anna Seigal and Bernd Sturmfels

Abstract

The real rank two locus of an algebraic variety is the closure of the union of all secant lines spanned by real points. We seek a semi-algebraic description of this set. Its algebraic boundary consists of the tangential variety and the edge variety. Our study of Segre and Veronese varieties yields a characterization of tensors of real rank two.

1 Introduction

Low-rank approximation of tensors is a fundamental problem in applied mathematics [3, 6]. We here approach this problem from the perspective of real algebraic geometry. Our goal is to give an exact semi-algebraic description of the set of tensors of real rank two and to characterize its boundary. This complements the results on tensors of non-negative rank two presented in [1], and it offers a generalization to the setting of arbitrary varieties, following [2].

A familiar example is that of $2 \times 2 \times 2$ -tensors (x_{ijk}) with real entries. Such a tensor lies in the closure of the real rank two tensors if and only if the *hyperdeterminant* is non-negative:

$$\begin{aligned} & x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{011}^2 x_{100}^2 + 4x_{000}x_{011}x_{101}x_{110} + 4x_{001}x_{010}x_{100}x_{111} \\ & - 2x_{000}x_{001}x_{110}x_{111} - 2x_{000}x_{010}x_{101}x_{111} - 2x_{000}x_{011}x_{100}x_{111} \\ & - 2x_{001}x_{010}x_{101}x_{110} - 2x_{001}x_{011}x_{100}x_{110} - 2x_{010}x_{011}x_{100}x_{101} \geq 0. \end{aligned} \quad (1)$$

If this inequality does not hold then the tensor has rank two over \mathbb{C} but rank three over \mathbb{R} .

To understand this example geometrically, consider the *Segre variety* $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, i.e. the set of rank one tensors, regarded as points in the projective space $\mathbb{P}^7 = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. The hyperdeterminant defines a quartic hypersurface $\tau(X)$ in \mathbb{P}^7 . The real projective space $\mathbb{P}_{\mathbb{R}}^7$ is divided into two connected components by its real points $\tau(X)_{\mathbb{R}}$. One of the two connected components is the locus $\rho(X)$ that comprises the real rank two tensors.

This paper views real rank in a general geometric framework, studied recently by Blekherman and Sinn [2]. Let X be an irreducible variety in a complex projective space \mathbb{P}^N that is defined over \mathbb{R} and whose set $X_{\mathbb{R}} = X \cap \mathbb{P}_{\mathbb{R}}^N$ of real points is Zariski dense in X . The *secant variety* $\sigma(X)$ is the closure of the set of points in \mathbb{P}^N that lie on a line spanned by two points in X . The *tangential variety* $\tau(X)$ is a subvariety of the secant variety. Namely, $\tau(X)$ is the closure of the set of points in \mathbb{P}^N that lie on a tangent line to X at a smooth point.

Our object of interest is the *real rank two locus* $\rho(X)$. This is a semi-algebraic set in the real projective space $\mathbb{P}_{\mathbb{R}}^N$. We define $\rho(X)$ as the closure of the set of points that lie on a line

spanned by two points in $X_{\mathbb{R}}$. Our hypotheses ensure that $\rho(X)$ is Zariski dense in $\sigma(X)$. The inclusion of the closed set $\rho(X)$ in the real secant variety $\sigma(X)_{\mathbb{R}}$ is usually strict. The difference consists of points of X -rank two whose real X -rank exceeds two.

Two varieties most relevant for applications are the *Segre variety* $X = \text{Seg}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1})$ and the *Veronese variety* $X = \nu_d(\mathbb{P}^{n-1})$. The ambient dimensions are $N = n_1 \cdots n_d - 1$ and $N = \binom{n+d-1}{d} - 1$, and X consists of (symmetric) tensors of rank one. The secant variety $\sigma(X)$ is the closure of the set of tensors of complex rank two, and $\sigma(X)_{\mathbb{R}}$ is the set of real points of that complex projective variety. The real rank two locus $\rho(X)$ is the closure of the tensors of real rank two. This is a subset of $\sigma(X)_{\mathbb{R}}$. The containment is strict when $d \geq 3$.

It is instructive to examine the case of $3 \times 2 \times 2$ -tensors. The secant variety $\sigma(X)$ has dimension 9 in \mathbb{P}^{11} . It consists of all tensors whose 3×4 matrix flattening satisfies

$$\text{rank} \begin{pmatrix} x_{000} & x_{001} & x_{010} & x_{011} \\ x_{100} & x_{101} & x_{110} & x_{111} \\ x_{200} & x_{201} & x_{210} & x_{211} \end{pmatrix} \leq 2. \quad (2)$$

The tangential variety $\tau(X)$ has codimension one in $\sigma(X)$. The ideal of $\tau(X)$ is generated by the 3×3 -minors of (2) and six hyperdeterminantal quartics. The set difference $\sigma(X)_{\mathbb{R}} \setminus \tau(X)_{\mathbb{R}}$ is disconnected. The topological closure of one of its connected components is the real rank two locus $\rho(X)$. Theorem 4.5 says that $\rho(X)$ is characterized by three inequalities like (1).

This article makes the following contributions. In Section 2 we determine the algebraic boundary of the real rank two locus $\rho(X)$, and we characterize boundary points that can be selected by Euclidean distance optimization. These results (Theorems 2.1 and 2.2) are for general varieties X . Section 3 offers a detailed study of the case when X is a space curve. Section 4 is devoted to the usual setting of tensors, when X is a Segre or Veronese variety. The real rank two locus for tensors is characterized by hyperdeterminantal inequalities (Theorem 4.5) and its algebraic boundary is given by the tangential variety (Theorem 4.3). In Section 5 we apply [13] to derive explicit equations (in Corollary 5.4) for that boundary when X is the Veronese. We also characterize symmetric $2 \times 2 \times \cdots \times 2$ -tensors of real rank two.

Our results here lay the geometric foundations for a subsequent paper that studies numerical algorithms for finding best real border rank two approximations of a given tensor.

2 Projective Varieties

We fix an irreducible real projective variety $X \subset \mathbb{P}^N$ whose set of real points $X_{\mathbb{R}}$ is Zariski dense in X . The tangential variety $\tau(X)$ is contained in the secant variety $\sigma(X)$. If the inclusion $\tau(X) \subset \sigma(X)$ is strict then both varieties have the expected dimensions:

$$\dim(\sigma(X)) = 2 \cdot \dim(X) + 1 \quad \text{and} \quad \dim(\tau(X)) = 2 \cdot \dim(X). \quad (3)$$

This is Theorem 1.4 in Zak's book [18]. If $\tau(X) = \sigma(X)$ then the variety X is called *defective*. Otherwise, the equalities in (3) hold, and we say that X is *non-defective*.

We write $\hat{X} \subset \mathbb{C}^{N+1}$ for the affine cone over X . The X -rank of a vector x in \mathbb{C}^{N+1} is the smallest r such that $x = x_1 + \cdots + x_r$ with x_1, \dots, x_r in \hat{X} , and analogously for points

x in \mathbb{P}^N . If x is real then its *real X -rank* is the smallest r such that $x = x_1 + \cdots + x_r$ with x_1, \dots, x_r in $\hat{X}_{\mathbb{R}}$. The loci of X -rank $\leq r$ and real X -rank $\leq r$ are typically not closed. We define the *X -border rank* and *real X -border rank* by passing to the closure of these loci. The secant variety $\sigma(X)$ consists of points of X -border rank ≤ 2 . The real rank two locus $\rho(X)$ consists of points of real X -border rank ≤ 2 . The latter is Zariski dense in the former.

The *real rank two boundary* $\partial(\rho(X))$ is the set $\rho(X)$ minus the relative interior of that set. This is a semi-algebraic subset of $\mathbb{P}_{\mathbb{R}}^N$. We note that $\partial(\rho(X))$ equals the topological boundary of $\rho(X)$, as discussed for similar settings in [8, §4] and [9, §5]. The Zariski closure of the set $\partial(\rho(X))$ in \mathbb{P}^N is denoted $\partial_{\text{alg}}(\rho(X))$ and is called the *algebraic real rank two boundary* of X . This projective variety has codimension one in $\sigma(X)$. Our aim is to describe it.

We need the following definitions. Let p and q be distinct smooth points on X whose corresponding tangent spaces $T_p(X)$ and $T_q(X)$ intersect in \mathbb{P}^N . The secant line spanned by such p and q is called an *edge* of X . The *edge variety* $\epsilon(X)$ is the closure in \mathbb{P}^N of the union of all edges of X . If $\dim(X) = (N-1)/2$ then the edge variety $\epsilon(X)$ is usually a hypersurface in $\sigma(X) = \mathbb{P}^N$. That hypersurface is the variety $(X^{[2]})^*$ in [15], and it plays an important role in convex algebraic geometry. For curves X in \mathbb{P}^3 , it is the edge surface in [16].

Theorem 2.1. *Let X be a non-defective variety in \mathbb{P}^N whose real points are Zariski dense. If the algebraic real rank two boundary of X is non-empty then it is a variety of pure codimension one inside the secant variety $\sigma(X)$. Its irreducible components arise from the tangential variety and the edge variety. In symbols, we have the equi-dimensional inclusion*

$$\partial_{\text{alg}}(\rho(X)) \subseteq \tau(X) \cup \epsilon(X). \quad (4)$$

The hypothesis that X is non-defective is essential for this theorem. For instance, if X is a plane curve in \mathbb{P}^2 then X is defective. Blekherman and Sinn [2, §3] showed that $\partial_{\text{alg}}(\rho(X))$ is a union of flex lines, provided it is non-empty. Such flex lines are not covered by (4).

The tangential variety $\tau(X)$ is always irreducible when X is irreducible. However, the edge variety $\epsilon(X)$ may be reducible even when X is irreducible. For instance, this happens when X is the curve obtained by intersecting two quadratic surfaces in \mathbb{P}^3 ; see [16, Example 2.3]. Therefore, it is possible that $\partial_{\text{alg}}(\rho(X))$ has more than two irreducible components.

Proof of Theorem 2.1. Recall from (3) that $\dim(\sigma(X)) = 2 \cdot \dim(X) + 1$. Hence, for a general real point u on the secant variety $\sigma(X)$, there are only finitely many pairs $\{v_1, w_1\}, \{v_2, w_2\}, \dots, \{v_k, w_k\}$ of points on X such that the line spanned by v_i and w_i contains u . The $2k$ points can be expressed locally as algebraic functions of u , by the Implicit Function Theorem. The point $u \in \sigma(X)_{\mathbb{R}}$ lies in $\rho(X)$ if at least one of these pairs $\{v_i, w_i\}$ consists of two real points, and it lies outside $\rho(X)$ if none of the pairs $\{v_i, w_i\}$ are real. By our assumption that the left hand side of (4) is non-empty, both cases are possible for X .

Consider a general real curve that passes through the boundary $\partial(\rho(X))$ at a point u^* , and follow the k point pairs along that curve. Precisely one of two scenarios will happen:

Case 1: A pair $\{v_i, w_i\}$ of real points merges into a single point on X and then transitions to a pair of conjugate complex points. As that transition occurs, the secant line degenerates to a tangent line. Hence the corresponding point u^* lies in the tangential variety $\tau(X)$.

Case 2: Two real pairs $\{v_i, w_i\}$ and $\{v_j, w_j\}$ come together, in the sense that v_i and v_j converge to a point $v \in X$ while w_i and w_j converge to another point $w \in X$. If this happens then the secant line spanned by v and w must be an edge, i.e. the tangent spaces $T_v(X)$ and $T_w(X)$ meet non-transversally. The corresponding point u^* lies in the edge variety $\epsilon(X)$.

Our argument above shows that each point in a dense subset of $\partial(\rho(X))$ lies either in $\tau(X)$ or in $\epsilon(X)$. Since these two sets are Zariski-closed in \mathbb{P}^N , it follows that the algebraic real rank two boundary $\partial_{\text{alg}}(\rho(X))$ is contained in their union $\tau(X) \cup \epsilon(X)$. \square

The present article was motivated by the following optimization problem:

Given data $u \in \mathbb{R}^{N+1}$, find the point u^* in the real rank two locus $\rho(X)$ that is closest to u .

Here and in what follows we identify projective varieties in \mathbb{P}^N and their affine cones in \mathbb{R}^{N+1} . The term “closest” refers to either the Euclidean norm or a weighted Euclidean norm as in [4, 8]. The algebraic complexity of this problem is measured by the *Euclidean distance degree* (ED degree). A priori, five scenarios are conceivable for random data u :

- (a) u^* is the point in $\sigma(X)_{\mathbb{R}}$ that is closest to u , and it is a smooth point of $\sigma(X)$.
- (b) u^* is the point in $X_{\mathbb{R}}$ that is closest to u ; in particular, it is a singular point of $\sigma(X)$.
- (c) u^* is the point in the singular locus of $\sigma(X)_{\mathbb{R}}$ that is closest to u , but it is not in X .
- (d) u^* is the point in $\tau(X)_{\mathbb{R}}$ that is closest to u .
- (e) u^* is the point in $\epsilon(X)_{\mathbb{R}}$ that is closest to u .

The solutions u^* in cases (d) and (e) are not critical for the distance function on $\sigma(X)$. The following theorem shows that case (b) cannot happen. This was proven for tensors by Stegeman and Friedland [17, Lemma 3.4]. We generalize their result to arbitrary varieties.

Theorem 2.2. *Suppose that \hat{X} does not lie on a hyperplane in \mathbb{R}^{N+1} . Let $u \in \mathbb{R}^{N+1}$ be a data point of real \hat{X} -border rank bigger than r and $u^* \in \mathbb{R}^{N+1}$ its best approximation of real \hat{X} -border rank at most r . Then the real \hat{X} -border rank of u^* is exactly r , not smaller.*

The best approximation is taken with respect to a weighted Euclidean distance on \mathbb{R}^{N+1} where all weights are strictly positive. The impossibility of case (b) is Theorem 2.2 for $r = 2$.

Proof. We begin with the case $r = 1$. Then $u \notin \hat{X}$ and we wish to show that u^* is non-zero. By our assumption, there exists a point x in the affine cone \hat{X} that is not in the hyperplane perpendicular to u . That is, $\langle u, x \rangle \neq 0$, where the inner product comes from our choice of norm. The point $\frac{\langle u, x \rangle}{\langle x, x \rangle} x$ lies in \hat{X} , and its squared distance to u is

$$\begin{aligned} \left\| \frac{\langle u, x \rangle}{\langle x, x \rangle} x - u \right\|^2 &= \left\langle \frac{\langle u, x \rangle}{\langle x, x \rangle} x - u, \frac{\langle u, x \rangle}{\langle x, x \rangle} x - u \right\rangle \\ &= \left(\frac{\langle u, x \rangle}{\langle x, x \rangle} \right)^2 \langle x, x \rangle - 2 \frac{\langle u, x \rangle}{\langle x, x \rangle} \langle u, x \rangle + \langle u, u \rangle = \langle u, u \rangle - \frac{\langle u, x \rangle^2}{\langle x, x \rangle}. \end{aligned}$$

This is strictly smaller than $\|u - 0\|^2 = \langle u, u \rangle$, so the closest point to u on \hat{X} is non-zero.

We now suppose that $r \geq 2$ and let u^* be the best approximation to u among points of real X -border rank at most r . Suppose that u^* has real \hat{X} -border rank strictly less than r . The point $v = u - u^*$ is non-zero. Its best real X -rank one approximation v^* is also non-zero. When $v \notin \hat{X}$, we use the previous paragraph to see this; otherwise $v^* = v \neq 0$. The point $u^* + v^*$ still has real \hat{X} -border rank at most r , and it is closer to u than u^* , since

$$\|u - (u^* + v^*)\| = \|v - v^*\| < \|v - 0\| = \|v\| = \|u - u^*\|.$$

Hence the best approximation to u cannot have real \hat{X} -border rank strictly less than r . \square

We have shown that case (b) cannot happen for best approximation by $\rho(X)$. All of the other four cases (a), (c), (d) and (e) are possible. Case (a) is the usual best real rank two approximation and it occurs frequently. Cases (d) and (e) occur for the curve in Example 3.4. We close this section by showing that case (c) occurs for rank two tensor approximation.

Example 2.3. Let $N = 26$ and fix $X = \text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$. According to [10, Cor. 7.17], the singular locus of $\sigma(X)$ has three irreducible components $\mathbb{P}^2 \times \sigma(\text{Seg}(\mathbb{P}^2 \times \mathbb{P}^2))$. These parametrize tensors $v \otimes M$, where $v \in \mathbb{R}^3$ and M is a 3×3 -matrix of rank two. Consider a data point $U = v \otimes M'$ where M' is a general real 3×3 -matrix. Let M be the best rank two approximation of M' . Then $U^* = v \otimes M$ is the unique best approximation to U in $\rho(X)$.

3 Space Curves

Blekherman and Sinn [2, §3] characterized the real rank two locus $\rho(X)$ for a curve X in the plane \mathbb{P}^2 . In this section we examine the case when X is a curve in \mathbb{P}^3 . We assume that X does not lie in a plane and that $X_{\mathbb{R}}$ is Zariski dense in X . The real X -rank of a general point $u \in \mathbb{P}_{\mathbb{R}}^3$ is either two or more, depending on whether the plane curve obtained by projecting X from u has real singularities or not. Specifically, any node on the projected curve corresponds to a line spanned by two real points on X that passes through u .

Remark 3.1. *The locus $\mathbb{P}_{\mathbb{R}}^3 \setminus \rho(X)$ of real X -border rank ≥ 3 consists of viewpoints u of crossing-free linear projections of $X_{\mathbb{R}}$. In particular, if $X_{\mathbb{R}}$ is a knot or link then $\rho(X) = \mathbb{P}_{\mathbb{R}}^3$.*

We use classical geometry to describe the transition between real ranks two and three. Let $u \in \mathbb{P}_{\mathbb{R}}^3$ and consider the plane curve in $\mathbb{P}_{\mathbb{R}}^2$ obtained by projecting X from the center u . If u has real X -rank two then that plane curve has a *crunode* (ordinary real double point). As u moves through space and transitions from real X -rank two to real X -rank three then that last crunode disappears. If the transition occurs via $\tau(X)$ then the intermediate singularity of the projected curve is a cusp. If it occurs via $\epsilon(X)$ then that singularity is a *tacnode*.

Figure 1 shows the transitions as the viewpoint u crosses the tangential surface $\tau(X)$ and the edge surface $\epsilon(X)$ respectively. These are two of the three classical *Reidemeister moves*. Transitions via the third Reidemeister move do not cause a change in real X -rank.

The edge surface $\epsilon(X)$ plays a prominent role in convex algebraic geometry. As shown in [16], it represents the non-linear part in the boundary of the convex hull of $X_{\mathbb{R}}$. See [16, Figures 1 and 2]. In this section we focus on rational curves. This allows us to use the methods in [16, Section 3]. We have the following result about the real rank two boundary.

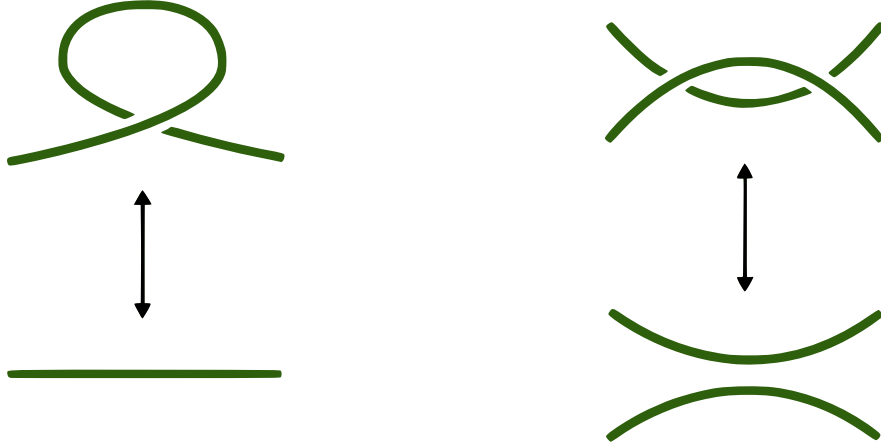


Figure 1: The viewpoint u crosses the tangential surface (left) or the edge surface (right).

Proposition 3.2. *There exist rational curves X_1, X_2, X_3 and X_4 in \mathbb{P}^3 such that*

$$\partial_{\text{alg}}(X_1) = \tau(X_1), \quad \partial_{\text{alg}}(X_2) = \epsilon(X_2) \cup \tau(X_2), \quad \partial_{\text{alg}}(X_3) = \emptyset, \quad \text{and} \quad \partial_{\text{alg}}(X_4) = \epsilon(X_4).$$

Proof. By Theorem 4.3, the twisted cubic curve in Example 4.10 can serve as the curve X_1 . The quartic curve in Example 3.4 serves as X_2 . For X_3 we take the Morton curve discussed in [16, Example 4.4]. This is rational of degree six and forms a trefoil knot [16, Figure 3].

Rational curves X_4 in $\mathbb{P}_{\mathbb{R}}^3$ with $\partial_{\text{alg}}(X_4) = \epsilon(X_4)$ are a bit harder to find. We constructed a piecewise-linear connected curve that has this property. Four views of that space curve are shown in Figure 2. There are relatively few viewpoints from which the curve has no crossings. From such positions, crossings are always gained in pairs, via transition along edges, as shown on the right in Figure 1. The existence of a rational algebraic curve X_4 with the same property can now be concluded from the Weierstrass Approximation Theorem. \square

In what follows we review the techniques in [16, pages 7-9], and we show how they can be adapted for computing rank two decompositions. Suppose that X is a rational curve of

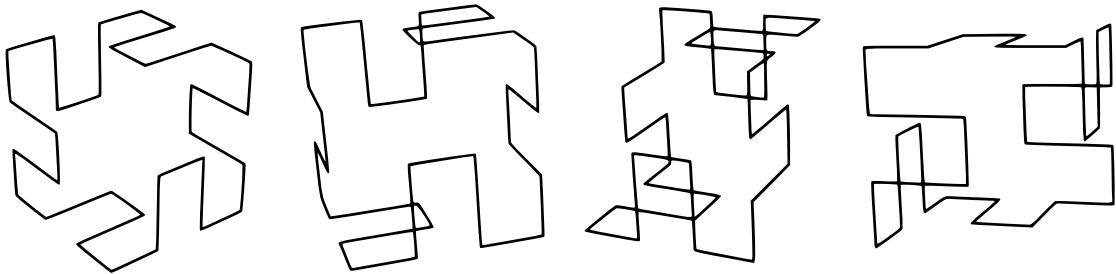


Figure 2: The space curve X_4 in Proposition 3.2, as seen from four different angles.

degree d in \mathbb{P}^3 . Note that $\sigma(X) = \mathbb{P}^3$. The curve X has a rational parametrization

$$\mathbb{P}^1 \rightarrow \mathbb{P}^3, (s : t) \mapsto (F_0(s, t) : F_1(s, t) : F_2(s, t) : F_3(s, t)).$$

Here, F_0, F_1, F_2, F_3 are binary forms of degree d . Fix two points $(s_1 : t_1)$ and $(s_2 : t_2)$ in \mathbb{P}^1 that represent two distinct points on the curve X . The secant line spanned by these two points in \mathbb{P}^3 has *Plücker coordinates* $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$ given by

$$p_{ij} = \frac{F_i(s_1, t_1)F_j(s_2, t_2) - F_i(s_2, t_2)F_j(s_1, t_1)}{s_1 t_2 - s_2 t_1} \quad \text{for } 0 \leq i < j \leq 3. \quad (5)$$

We represent the unordered pair $\{(s_1 : t_1), (s_2 : t_2)\}$ of points in \mathbb{P}^1 by the binary quadric

$$ax^2 + bxy + cy^2 = (s_1 x + t_1 y)(s_2 x + t_2 y). \quad (6)$$

Each p_{ij} is invariant under swapping $(s_1 : t_1)$ and $(s_2 : t_2)$. Hence we can write p_{ij} as a homogeneous polynomial of degree $d - 1$ in (a, b, c) . The resulting formulas define a rational map from $\mathbb{P}^2 = \text{Sym}_2(\mathbb{P}^1)$ into the Grassmannian of lines $\text{Gr}(1, \mathbb{P}^3)$. This parametrizes the secant lines. The points $(w : x : y : z) \in \mathbb{P}^3$ on a particular secant line are the solutions of

$$\begin{pmatrix} 0 & p_{23} & -p_{13} & p_{12} \\ -p_{23} & 0 & p_{03} & -p_{02} \\ p_{13} & -p_{03} & 0 & p_{01} \\ -p_{12} & p_{02} & -p_{01} & 0 \end{pmatrix} \cdot \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (7)$$

This is a system of equations of bidegree $(d - 1, 1)$ in $((a, b, c), (w, x, y, z))$. They define a threefold in $\mathbb{P}^2 \times \mathbb{P}^3$. The fiber of a data vector (w, x, y, z) in \mathbb{R}^4 , under the map that projects the threefold onto the second factor \mathbb{P}^3 , gives its X -rank two decompositions. To be concrete, given real numbers w, x, y, z , we plug them into the system (7). This yields four homogeneous equations of degree $d - 1$ in three unknowns a, b, c , of which two are linearly independent. The X -rank two decomposition defined by a triple (a, b, c) is then obtained from equation (6). The reality of the X -rank two decomposition is determined as follows.

Proposition 3.3. *The point $u = (w : x : y : z)$ has real X -rank ≤ 2 if and only if the system (7) has a real solution $(a : b : c)$ such that the matrix in (7) is non-zero and the discriminant $b^2 - 4ac$ of the quadric (6) is positive. Such points $(a : b : c) \in \mathbb{P}_{\mathbb{R}}^2$ are in bijection with lines in $\mathbb{P}_{\mathbb{R}}^3$ that pass through u and meet the curve X in two real points.*

The boundary $\partial(\rho(X))$ marks the transition between systems (7) that admit solutions as described in Proposition 3.3 and those that do not. The discriminantal surface in \mathbb{P}^3 that separates real X -rank ≤ 2 from real X -rank ≥ 3 can have components contributed by both the tangential variety $\tau(X)$ and the edge surface $\epsilon(X)$. These are ruled surfaces. The lines in the rulings are represented by curves in the plane \mathbb{P}^2 with coordinates $(a : b : c)$. To obtain the surface from each curve, we compute its image under the correspondence (7).

The first relevant curve is the conic $b^2 - 4ac$. This encodes binary quadrics (6) with a double root, so its image in \mathbb{P}^3 is the tangential surface $\tau(X)$. The second relevant curve has degree $2(d - 3)$. Its defining polynomial $\Phi(a, b, c)$ was constructed in [16, equation (3.6)]. The image of the curve $\Phi = 0$ under the correspondence (7) is the edge surface $\epsilon(X)$ in \mathbb{P}^3 .

Example 3.4. Let $d = 4$ and fix the smooth monomial curve X in \mathbb{P}^3 with parametrization

$$F_0 = s^4, \quad F_1 = s^3t, \quad F_2 = st^3, \quad F_3 = t^4. \quad (8)$$

The parametrization (5) of the secant lines of X in terms of Plücker coordinates is

$$\begin{array}{lll} p_{01} & = & a^3, \\ p_{12} & = & abc, \end{array} \quad \begin{array}{lll} p_{02} & = & a(b^2 - ac), \\ p_{13} & = & c(b^2 - ac), \end{array} \quad \begin{array}{lll} p_{03} & = & b(b^2 - 2ac), \\ p_{23} & = & c^3. \end{array}$$

The secant correspondence in $\mathbb{P}^2 \times \mathbb{P}^3$ is obtained by substituting these expressions into (7), and saturating with respect to the p_{ij} . Its map onto \mathbb{P}^3 has degree three. A general point $u = (w : x : y : z)$ in \mathbb{P}^3 lies on three secants, each represented by a point $(a : b : c)$. The semi-algebraic set $\rho(X)$ consists of points where at least one of the three secants is real and meets X in two real points. Algebraically, we desire that a, b, c are real and satisfy $b^2 \geq 4ac$.

The tangential surface $\tau(X)$ has degree 6. We compute its defining equation as follows. First add $b^2 - 4ac$ to the ideal in (7), then saturate by the entries of the skew-symmetric 4×4 -matrix, and finally eliminate the unknowns a, b, c . The result is the polynomial

$$16x^3y^3 - 27w^2y^4 + 6wx^2y^2z - 27x^4z^2 + 48w^2xyz^2 - 16w^3z^3. \quad (9)$$

The edge surface $\epsilon(X)$ has degree 6 as well. Following [16, equation (3.6)], it is encoded by the plane quadric $\Phi(a, b, c) = b^2 + 2ac$. The same elimination process yields the polynomial

$$32x^3y^3 - 27w^2y^4 - 6wx^2y^2z - 27x^4z^2 + 24w^2xyz^2 + 4w^3z^3. \quad (10)$$

The ruled sextic surfaces (9) and (10) divide $\mathbb{P}_{\mathbb{R}}^3$ into various connected components. The real rank two locus $\rho(X)$ is the union of the components whose points obey Proposition 3.3.

We claim that $\rho(X)$ is a proper subset of $\mathbb{P}_{\mathbb{R}}^3$ and that both the edge surface $\epsilon(X)$ and the tangential surface $\tau(X)$ contribute to the real rank boundary $\partial(\rho(X))$. To prove this claim, we consider the line segment in $\mathbb{P}_{\mathbb{R}}^3$ whose points $u(t)$ are given by the parametrization

$$w = 84 - 74t, \quad x = 13 + 59t, \quad y = 62 - 19t, \quad z = -38 - 10t.$$

Here t is a real parameter that runs from 0 to 1. By substituting into (9) and (10) respectively, we find that the line segment crosses the tangential surface $\tau(X)$ twice, namely when

$$t_1 = 0.41616468475415957221 \quad \text{and} \quad t_3 = 0.64786245578375696533.$$

It also crosses the edge surface $\epsilon(X)$ twice, namely at the points $u(t_2)$ and $u(t_4)$ given by

$$t_2 = 0.50734775284175190900 \quad \text{and} \quad t_4 = 0.81105706603104911043.$$

These four parameter values divides the line segment into five smaller segments on which the corresponding secant lines of X have constant real behavior. Computations reveal:

- For $0 < t < t_1$, the real X -rank of $u(t)$ is 3. One of the three complex secant lines is real but it does not meet the curve X in real points.

- For $t_1 < t < t_2$, the real X -rank of $u(t)$ is 2. One of the three complex secant lines is real and it meets the curve X in two real points.
- For $t_2 < t < t_3$, the real X -rank of $u(t)$ is 2. All the three complex secant lines are real and they all meet the curve X in two real points.
- For $t_3 < t < t_4$, the real X -rank of $u(t)$ is 2. All the three complex secant lines are real but only two of them meet the curve X in two real points.
- For $t_4 < t < 1$ the real X -rank of $u(t)$ is 3. One of the three complex secant lines is real but it does not meet the curve X in real points.

This verifies that both of the transitions depicted in Figure 1 do occur along this line segment. At $t = t_1$ the real X -rank changes by crossing the tangential surface, and at $t = t_4$ it changes by crossing the edge surface. Additional crossings of the two boundary surfaces take place at $t = t_3$ and at $t = t_2$, but these do not change the real X -rank of $u(t)$.

We finally note that both of the two scenarios (d) and (e) for rank two approximation, discussed prior to Theorem 2.2, are realized for X along this line segment. Namely, for sufficiently small $\epsilon > 0$, we obtain (d) for $u = u(t_1 - \epsilon)$, and we obtain (e) for $u = u(t_4 + \epsilon)$.

4 Tensors and their Hyperdeterminants

The varieties X whose ranks are most relevant for applications are the Segre variety and the Veronese variety. When studying tensors of format $n_1 \times n_2 \times \cdots \times n_d$, we set $N = n_1 n_2 \cdots n_d - 1$ and $X \subset \mathbb{P}^N$ is the *Segre variety* whose points are tensors of rank one. When studying symmetric tensors of format $n \times n \times \cdots \times n$ with d factors, we set $N = \binom{n+d-1}{d} - 1$ and $X \subset \mathbb{P}^N$ is the *Veronese variety* whose points are symmetric tensors of rank one. These two classical varieties X are non-defective provided $d \geq 3$. We exclude the case $d = 2$ because the corresponding varieties of rank one matrices are defective.

For any variety X as before, the degree of the natural parametrization of its secant variety $\sigma(X)$ gives the number of rank two decompositions of a generic point. It is the integer k in the proof of Theorem 2.1. If the parametrization is birational ($k = 1$) then $\sigma(X)$ is said to be *identifiable*. If the secant variety $\sigma(X)$ is identifiable then there is no edge variety $\epsilon(X)$.

Remark 4.1. It is natural to wonder whether $\tau(X)_{\mathbb{R}}$ is always contained in the real rank two locus $\rho(X)$. Furthermore, if $\sigma(X)$ is identifiable, then $\tau(X)_{\mathbb{R}} \subseteq \partial(\rho(X))$ seems plausible. This would be true if every transition through $\tau(X)_{\mathbb{R}}$ were as in Case 1 of Theorem 2.1. However, this may be false. For instance, consider a smooth space curve X as in Section 3. The tangential surface $\tau(X)_{\mathbb{R}}$ can look locally like a *Whitney umbrella*. It might have lower-dimensional real pieces that protrude into the interior of $\rho(X)$ or its complement. If $\sigma(X)$ is not identifiable then the interior of $\rho(X)$ can contain a region of $\tau(X)_{\mathbb{R}}$ that is Zariski dense in $\tau(X)$. The point $u(t_3)$ in Example 3.4 lies inside such a region of the tangential surface.

We now focus on the case of tensors, where X is a Segre or Veronese variety. Here, the secant variety is usually identifiable, and Remark 4.1 can be strengthened as follows.

Lemma 4.2. *Let X be a Segre or Veronese variety with $d \geq 3$. Then the real tangential variety is contained in the boundary of the real rank two locus; in symbols, $\tau(X)_{\mathbb{R}} \subseteq \partial(\rho(X))$.*

Proof. Let T be a real point in $\tau(X)$. It is expressible as a sum of d rank one tensors,

$$T = t_1 \otimes s_2 \otimes \cdots \otimes s_d + s_1 \otimes t_2 \otimes s_3 \otimes \cdots \otimes s_d + \cdots + s_1 \otimes \cdots \otimes s_{d-2} \otimes t_{d-1} \otimes s_d + s_1 \otimes \cdots \otimes s_{d-1} \otimes t_d,$$

where we omit the subscripts for the Veronese case. This representation follows from [7, §8.1.1]. Direct computations show that there exists a sequence of tensors $a_\epsilon \xrightarrow{\epsilon \rightarrow 0} T$ where each a_ϵ lies on the secant line spanned by $(s_1 + \epsilon t_1) \otimes (s_2 + \epsilon t_2) \otimes \cdots \otimes (s_d + \epsilon t_d)$ and $s_1 \otimes s_2 \otimes \cdots \otimes s_d$. There is also a sequence of real tensors $b_\epsilon \rightarrow T$ where each b_ϵ lies on the secant line spanned by $(s_1 + i\epsilon t_1) \otimes (s_2 + i\epsilon t_2) \otimes \cdots \otimes (s_d + i\epsilon t_d)$ and $(s_1 - i\epsilon t_1) \otimes (s_2 - i\epsilon t_2) \otimes \cdots \otimes (s_d - i\epsilon t_d)$. Here $i = \sqrt{-1}$. See Example 5.10 for the case when X is a rational normal curve. By Kruskal's Theorem [6, §3.2], these real (resp. complex) expressions for a_ϵ (resp. b_ϵ) are unique, provided three or more t_i are non-zero. Therefore, T is both a limit of tensors in $\rho(X)$ and a limit of tensors that are not in $\rho(X)$. Hence T lies in the boundary $\partial(\rho(X))$. It remains to consider the case when at most two t_i are non-zero. This case is similar, but other sequences α_ϵ and β_ϵ are used to approximate T from inside and outside $\rho(X)$. \square

We have the following characterization of the algebraic real rank two boundary for tensors.

Theorem 4.3. *Let X be the Segre variety (resp. the Veronese variety) whose points are d -dimensional tensors (resp. symmetric tensors) of rank one. If $d \geq 3$ then the algebraic real rank two boundary of X is non-empty and equals the tangential variety of X . In symbols,*

$$\partial_{\text{alg}}(\rho(X)) = \tau(X).$$

Proof. The secant variety $\sigma(X)$ is identifiable, since Kruskal's Theorem holds generically for rank two tensors. Therefore, $\epsilon(X)$ does not exist. To prove the theorem, we must exclude the possibility $\partial_{\text{alg}}(\rho(X)) = \emptyset$. By taking sums of complex conjugate pairs of points on the affine cone \hat{X} , one creates many tensors that lie in $\sigma(X)_{\mathbb{R}}$ but not in $\rho(X)$. Hence the rank two locus $\rho(X)$ has a non-empty boundary inside $\sigma(X)_{\mathbb{R}}$, and the algebraic boundary $\partial_{\text{alg}}(\rho(X))$ is a non-empty hypersurface in $\sigma(X)$. That hypersurface is contained in the irreducible hypersurface $\tau(X)$, by Theorem 2.1. This implies that they are equal. \square

Remark 4.4. Identifiability of $\sigma(X)$, for X the Segre or Veronese variety, can be used to show that a symmetric tensor of (real) symmetric rank two also has (real) rank two. This is *Comon's Conjecture* in the symmetric rank two case, see e.g. [5]. If a real tensor T has symmetric real rank two then it has the unique decomposition into rank one tensors:

$$T = a^{\otimes d} + b^{\otimes d}.$$

The summands are real and symmetric. Identifiability of $\sigma(X)$, for X the Segre variety, means there is no other representation of T as a sum of two tensors of rank one.

We next derive the following general result concerning tensors T of arbitrary format $n_1 \times n_2 \times \cdots \times n_d$ where $d \geq 3$. A $2 \times 2 \times 2$ sub-tensor of T is obtained by fixing $d - 3$ of the indices and selecting pairs for the other three. We are interested in the hyperdeterminants of these sub-tensors. These are the $2 \times 2 \times 2$ *sub-hyperdeterminants* of T . Their number is

$$\frac{1}{8} \cdot n_1 n_2 n_3 \cdots n_d \cdot \sum_{1 \leq i < j < k \leq d} (n_i - 1)(n_j - 1)(n_k - 1). \quad (11)$$

Theorem 4.5. *A real tensor T has real border rank ≤ 2 if and only if all of its flattenings have rank ≤ 2 and all of its $2 \times 2 \times 2$ sub-hyperdeterminants are non-negative. If this holds then the real rank of T is exactly two if at least one of the flattenings of T has rank two and at least one of the $2 \times 2 \times 2$ sub-hyperdeterminants of T is strictly positive.*

Proof. We begin with the only-if direction of the first statement. Let T have real border rank ≤ 2 . Then every $2 \times 2 \times 2$ sub-tensor T' has real border rank ≤ 2 . We can approximate T' by a sequence of tensors T'' that have real rank two. The entries t''_{ijk} of any tensor in the approximating sequence can be written as $t''_{ijk} = a_i b_j c_k + d_i e_j f_k$, where the parameters are real. With a computation one checks that the hyperdeterminant of T'' evaluates to

$$(a_1 d_2 - a_2 d_1)^2 (b_1 e_2 - b_2 e_1)^2 (c_1 f_2 - c_2 f_1)^2.$$

This quantity is non-negative since all parameters are real. By continuity, we conclude that all $2 \times 2 \times 2$ sub-hyperdeterminants of the original tensor T are non-negative.

For the if direction, suppose that T is a tensor in $\sigma(X)_{\mathbb{R}}$ whose $2 \times 2 \times 2$ sub-hyperdeterminants are all non-negative. The complex rank of T is either 1, 2 or ≥ 3 . If it is 1 then T is in the real Segre variety $X_{\mathbb{R}}$ and hence in $\rho(X)$. If T has complex rank ≥ 3 then it is in $\tau(X)_{\mathbb{R}} \setminus X$, and we deduce that $T \in \rho(X)$ from Lemma 4.2.

It remains to examine the case when T has complex rank two and real rank ≥ 3 . The tensor T lies on a real secant line, spanned by a pair of complex conjugate points in X . Consider any $2 \times 2 \times 2$ sub-tensor T' of T . We can write the entries t'_{ijk} of T' as

$$t'_{ijk} = (a_i + A_i \sqrt{-1})(b_j + B_j \sqrt{-1})(c_k + C_k \sqrt{-1}) + (a_i - A_i \sqrt{-1})(b_j - B_j \sqrt{-1})(c_k - C_k \sqrt{-1}),$$

where the parameters a, b, c, A, B, C are real. One checks that the hyperdeterminant of T' is

$$-(a_1 A_2 - a_2 A_1)^2 \cdot (b_1 B_2 - b_2 B_1)^2 \cdot (c_1 C_2 - c_2 C_1)^2 \cdot 4^3. \quad (12)$$

This expression is non-positive since all parameters are real. Our hypothesis that all $2 \times 2 \times 2$ sub-hyperdeterminants are non-negative means they must all be zero.

The rank two representation of T involves pairs of vectors $\{a, A\} \subset \mathbb{R}^{n_1}$, $\{b, B\} \subset \mathbb{R}^{n_2}$, $\{c, C\} \subset \mathbb{R}^{n_3}, \dots$. Every $2 \times 2 \times 2$ sub-hyperdeterminant of T has the form in (12) and equates to zero. From this we conclude that, for all but two of the pairs $\{a, A\}, \{b, B\}, \{c, C\}, \dots$, the vectors in the pair are linearly dependent. If not, we could choose indices (i, j) from each vector pair for which the expression $a_i A_j - a_j A_i$ does not vanish, yielding a non-vanishing sub-hyperdeterminant. Hence T is the tensor product of a matrix with $d - 2$ vectors. This contradicts the hypothesis that T has real rank exceeding two.

If T is rank one, all flattenings have rank one and all $2 \times 2 \times 2$ sub-hyperdeterminants vanish. So if one flattening has rank two, or one sub-hyperdeterminant is strictly positive, the real rank of T must be at least two. To conclude the proof, it remains to consider tensors in $\rho(X)$ whose real rank exceeds two but are nonetheless there due to taking the closure.

Such tensors lie in $\partial(\rho(X))$, and hence in the tangential variety $\tau(X)$. We claim that all sub-hyperdeterminants vanish on $\tau(X)$. This is immediate in the base case $X = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in \mathbb{P}^7 , since $\tau(X)$ equals the vanishing locus of the hyperdeterminant. For larger tensor formats, the projection of the tangential variety to any $2 \times 2 \times 2$ sub-tensor is precisely that same tangential variety. Hence each $2 \times 2 \times 2$ sub-hyperdeterminant vanishes on $\tau(X)$, for Segre varieties X of arbitrary size. Thus, if a tensor has at least one flattening of rank two, and at least one sub-hyperdeterminant strictly positive, it has real rank exactly two. \square

Example 4.6. It is instructive to work through this proof for $2 \times 2 \times 2 \times 2$ -tensors T . If T has complex rank two and real rank ≥ 3 then its entries t_{ijkl} have the parametric representation

$$t_{ijkl} = (a_i + A_i\sqrt{-1})(b_j + B_j\sqrt{-1})(c_k + C_k\sqrt{-1})(d_l + D_l\sqrt{-1}) \\ + (a_i - A_i\sqrt{-1})(b_j - B_j\sqrt{-1})(c_k - C_k\sqrt{-1})(d_l - D_l\sqrt{-1}).$$

Suppose the eight $2 \times 2 \times 2$ sub-hyperdeterminants of T are all non-negative. They are

$$\begin{aligned} & -(a_0^2 + A_0^2)^2(b_0B_1 - b_1B_0)^2(c_0C_1 - c_1C_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(a_1^2 + A_1^2)^2(b_0B_1 - b_1B_0)^2(c_0C_1 - c_1C_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(b_0^2 + B_0^2)^2(a_0A_1 - a_1A_0)^2(c_0C_1 - c_1C_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(b_1^2 + B_1^2)^2(a_0A_1 - a_1A_0)^2(c_0C_1 - c_1C_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(c_0^2 + C_0^2)^2(a_0A_1 - a_1A_0)^2(b_0B_1 - b_1B_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(c_1^2 + C_1^2)^2(a_0A_1 - a_1A_0)^2(b_0B_1 - b_1B_0)^2(d_0D_1 - d_1D_0)^24^3, \\ & -(d_0^2 + D_0^2)^2(a_0A_1 - a_1A_0)^2(b_0B_1 - b_1B_0)^2(c_0C_1 - c_1C_0)^24^3, \\ & -(d_1^2 + D_1^2)^2(a_0A_1 - a_1A_0)^2(b_0B_1 - b_1B_0)^2(c_0C_1 - c_1C_0)^24^3. \end{aligned} \tag{13}$$

We note that the first factor does not appear in equation (12) because the fixed indices were subsumed into the expressions for one of the parameter pairs $\{a, A\}, \{b, B\}, \{c, C\}$.

It cannot be that a_0, A_0, a_1, A_1 are all zero, and similarly for the other letters. Hence

$$(a_0A_1 - a_1A_0)(b_0B_1 - b_1B_0)(c_0C_1 - c_1C_0) = (a_0A_1 - a_1A_0)(b_0B_1 - b_1B_0)(d_0D_1 - d_1D_0) = \\ (a_0A_1 - a_1A_0)(c_0C_1 - c_1C_0)(d_0D_1 - d_1D_0) = (b_0B_1 - b_1B_0)(c_0C_1 - c_1C_0)(d_0D_1 - d_1D_0) = 0.$$

Two of the four factors are zero. There are six cases. Up to relabeling, $a_0A_1 - a_1A_0 = b_0B_1 - b_1B_0 = 0$. This implies that $T = (a_0, a_1) \otimes (b_0, b_1) \otimes U$, where U is a 2×2 -matrix. Clearly U has real rank ≤ 2 . This shows that T has real rank ≤ 2 , the necessary contradiction.

We briefly discuss the implications of our inequality description of the real rank two locus $\rho(X)$ for its boundary $\partial(\rho(X))$. Here X is the Segre variety of rank one tensors. Let Hyp denote the variety consisting of tensors whose $2 \times 2 \times 2$ sub-hyperdeterminants are all zero.

Proposition 4.7. *The real rank two boundary $\partial(\rho(X))$ is a subset of the semi-algebraic set $\text{Hyp} \cap \rho(X)$. This containment is an equality for tensors of order $d = 3$ but strict for $d \geq 4$.*

Proof. We saw in Theorem 4.3 that $\partial(\rho(X))$ is contained in the tangential variety $\tau(X)$. So for the first claim, it suffices to show that each $2 \times 2 \times 2$ sub-hyperdeterminant vanishes on $\tau(X)$. This was shown in end of the proof of Theorem 4.5; see also [11, Theorem 1.3].

Suppose that $d = 3$ and $T = (t_{ijk})$ is an $n_1 \times n_2 \times n_3$ -tensor in $\text{Hyp} \cap \rho(X)$. If T lies in $\tau(X)$ then it is in the boundary $\partial(\rho(X))$, by Lemma 4.2. We may therefore assume that T has real rank ≤ 2 . So, its entries can be written as $t_{ijk} = a_i b_j c_k + d_i e_j f_k$. Since $T \in \text{Hyp}$, for all indices $1 \leq i_1 < i_2 \leq n_1$, $1 \leq j_1 < j_2 \leq n_2$ and $1 \leq k_1 < k_2 \leq n_3$, we have

$$(a_{i_1} d_{i_2} - a_{i_2} d_{i_1}) \cdot (b_{j_1} e_{j_2} - b_{j_2} e_{j_1}) \cdot (c_{k_1} f_{k_2} - c_{k_2} f_{k_1}) = 0.$$

This condition implies that either $\{a, d\}$ or $\{b, e\}$ or $\{c, f\}$ are linearly dependent. After relabeling and rescaling we may assume $a = d$. This implies $T = a \otimes ((b \otimes c) + (e \otimes f))$. This tensor lies in the tangential variety $\tau(X)$ of the Segre variety $X = \text{Seg}(\mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} \times \mathbb{P}^{n_3-1})$.

It remains to show that $\partial(\rho(X))$ is strictly contained in $\text{Hyp} \cap \rho(X)$ for $d \geq 4$. Consider $d = 4$ and $X = \text{Seg}((\mathbb{P}^1)^4)$. Let $\{e_1, e_2\}$ be the standard basis of \mathbb{R}^2 . The rank two tensor

$$T = e_1 \otimes e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2 \otimes e_2 \quad (14)$$

is in the relative interior of the real rank two locus $\rho(X)$. All eight $2 \times 2 \times 2$ sub-tensors have rank one, so the eight hyperdeterminants vanish. Hence T lies in $\text{Hyp} \cap \rho(X) \setminus \partial(\rho(X))$. This tensor can now be embedded into all larger formats, and we get the conclusion for $d \geq 4$. \square

Remark 4.8. The tensor (14) shows that the only-if direction in the second sentence of Theorem 4.5 does not hold. It has rank two but no $2 \times 2 \times 2$ hyperdeterminant is positive.

Remark 4.9. The number (11) of $2 \times 2 \times 2$ sub-hyperdeterminants for a tensor of format $n \times n \times \cdots \times n$ equals $\frac{1}{8} \binom{d}{3} n^d (n-1)^3$. If the tensor is symmetric then this number reduces to

$$\binom{n+d-4}{n-1} \binom{\binom{n}{2}+2}{3}. \quad (15)$$

Among these we only need hyperdeterminants whose expansion as in (13) is a sixth power like $(a_0 A_1 - a_1 A_0)^6$ times an extraneous factor $\prod_i (a_i^2 + A_i^2)^2$. That reduces the number to

$$\binom{n+d-4}{n-1} \binom{n}{2}. \quad (16)$$

Each of these symmetric hyperdeterminants looks like the quartic D in the next example.

Example 4.10. Let $n = 2, d = 3$. Here X is the twisted cubic curve in \mathbb{P}^3 . The tangential variety $\tau(X)$ is the quartic surface in $\sigma(X) = \mathbb{P}^3$ given by the discriminant of a binary cubic:

$$D = x_0^2 x_3^2 - 6x_0 x_1 x_2 x_3 - 3x_1^2 x_2^2 + 4x_1^3 x_3 + 4x_0 x_2^3 = \det \begin{pmatrix} x_0 & 2x_1 & x_2 & 0 \\ 0 & x_0 & 2x_1 & x_2 \\ x_1 & 2x_2 & x_3 & 0 \\ 0 & x_1 & 2x_2 & x_3 \end{pmatrix} \quad (17)$$

This is the $2 \times 2 \times 2$ hyperdeterminant (1) specialized to symmetric tensors [12, page 2]. Both numbers (15) and (16) are one. The real rank two locus $\rho(X)$ is the subset of $\mathbb{P}_{\mathbb{R}}^3$ defined by $D \geq 0$. For a study of hyperdeterminants of symmetric tensors we refer to Oeding [12].

5 The Tangential Variety of the Veronese

The variety $\sigma(X)$ of rank two tensors is defined by the 3×3 -minors of all flattenings [14]. Among the real points on that secant variety, the locus $\rho(X)$ is defined by the hyperdeterminantal inequalities in Theorem 4.5. Since the algebraic boundary of $\rho(X)$ is the tangential variety $\tau(X)$, one might think that $\tau(X)$ is obtained by setting the hyperdeterminants to zero. But this is false, as seen in Proposition 4.7. Oeding and Raicu [13, 14] showed that $\tau(X)$ is often defined by quadrics. In this section we focus on Veronese varieties, and we translate the representation-theoretic results from [13] into explicit quadrics. We close with examples that illustrate the findings in our paper for the rational normal curve $X = \nu_d(\mathbb{P}^1)$.

The following result is for tensors with $d \geq 3$. The variety X comprises rank one tensors, so it is the Segre variety $X = \text{Seg}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1})$ or the Veronese variety $X = \nu_d(\mathbb{P}^{n-1})$.

Theorem 5.1 (Oeding-Raicu [13], Raicu [14]). *The ideal of the secant variety $\sigma(X)$ is generated by the 3×3 -minors of the various flattenings of the tensor. For symmetric tensors, it suffices to take the 3×3 -minors of the most symmetric catalecticant matrix. The ideal of the tangential variety $\tau(X)$ is generated in degree at most four; the Schur modules of minimal generators are known explicitly. If $d \geq 5$ then quadrics suffice to generate the ideal of $\tau(X)$.*

The space of minimal generators of the prime ideals in question is a G -module, where $G = \text{SL}(n)$ if $X = \nu_d(\mathbb{P}^{n-1})$ and $G = \text{SL}(n_1) \times \cdots \times \text{SL}(n_d)$ if $X = \text{Seg}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1})$. The term *Schur module* refers to the irreducible representations that occur in these G -modules. We shall use basics from the representation theory of G , as in Landsberg's book [7].

Our aim is to extract explicit polynomials from the last two sentences in Theorem 5.1, for the case when $X = \nu_d(\mathbb{P}^{n-1})$ and $G = \text{SL}(n)$. The irreducible G -modules of degree d are indexed by partitions λ of d with at most n parts. The module for λ is denoted S_λ . It has a natural basis, labeled by semi-standard Young tableaux of shape λ filled with $\{1, 2, \dots, n\}$.

We shall present a basis for the space $I_2(\tau(X))$ of quadrics that vanish on $\tau(X)$. All such quadrics are minimal ideal generators, since $\tau(X)$ does not lie in a linear subspace of $\mathbb{P}^{\binom{n+d-1}{d}-1}$. Proposition 5.5 says that $I_2(\tau(X))$ usually defines $\tau(X)$ as a subvariety of $\sigma(X)$.

Fix an even positive integer k and consider the irreducible G -module $S_\lambda(\mathbb{C}^n)$ where λ is the partition $(2d-k, k)$. We draw λ as a shape with two rows, the first of length $2d-k$ and the second of length k . A basis of $S_\lambda(\mathbb{C}^n)$ is indexed by the semi-standard Young tableaux (SSYT) of shape λ filled with integers between 1 and n . The SSYT of shape λ are identified with pairs (μ, ν) of row vectors $\mu \in \{1, 2, \dots, n\}^{2d-k}$ and $\nu \in \{1, 2, \dots, n\}^k$ that satisfy

$$\begin{aligned} \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \mu_{k+1} \leq \cdots \leq \mu_{2d-k}, \\ \nu_1 \leq \nu_2 \leq \nu_3 \leq \cdots \leq \nu_k \quad \text{and} \quad \mu_i < \nu_i \text{ for } i = 1, 2, \dots, k. \end{aligned} \tag{18}$$

By the *Hook Length Formula*, the number of such SSYT of shape λ equals

$$\dim(S_\lambda(\mathbb{C}^n)) = \prod_{i=1}^k \frac{n-1+i}{2d+2-k-i} \cdot \prod_{i=k+1}^{2d-k} \frac{n-1+i}{2d+1-k-i} \cdot \prod_{j=1}^k \frac{n-2+j}{k+1-j}. \tag{19}$$

We realize $S_\lambda(\mathbb{C}^n)$ as a submodule of $(\mathbb{C}^n)^{\otimes 2d}$ by assigning to (μ, ν) with (18) the basis vector

$$(e_{\mu_1} \wedge e_{\nu_1}) \otimes (e_{\mu_2} \wedge e_{\nu_2}) \otimes \cdots \otimes (e_{\mu_k} \wedge e_{\nu_k}) \otimes [e_{\mu_{k+1}} e_{\mu_{k+2}} \cdots e_{\mu_{2d-k}}]. \quad (20)$$

Here e_1, \dots, e_n is the standard basis of \mathbb{C}^n , the symbol \wedge denotes antisymmetrization of the tensor product, and the expression $[\cdots]$ is the symmetrization of that tensor product.

We next translate the expression (20) into a quadratic polynomial in the $\binom{n+d-1}{d}$ homogeneous coordinates x_u on $\mathbb{P}^{\binom{n+d-1}{d}-1}$. This polynomial is supposed to vanish on $\tau(X)$. We write $(t_1 : t_2 : \cdots : t_n)$ for the homogeneous coordinates on \mathbb{P}^{n-1} . The parametrization of $\sigma(X)$ by pairs of points in the cone over the Veronese variety X can be written as follows:

$$\sum_{|u|=d} \binom{|u|}{u} x_u t^u = (a_1 t_1 + a_2 t_2 + \cdots + a_n t_n)^d + (b_1 t_1 + b_2 t_2 + \cdots + b_n t_n)^d. \quad (21)$$

We translate the expression (20) into the following polynomial in the $2n$ parameters:

$$\prod_{i=1}^k (a_{\mu_i} b_{\nu_i} - a_{\nu_i} b_{\mu_i}) \cdot \left(\sum a_{\mu_{j_1}} a_{\mu_{j_2}} \cdots a_{\mu_{j_{d-k}}} b_{\mu_{j_{d-k+1}}} b_{\mu_{j_{d-k+2}}} \cdots b_{\mu_{j_{2d-2k}}} \right), \quad (22)$$

where the sum is over permutations $(j_1, j_2, \dots, j_{2d-2k})$ of $\{k+1, k+2, \dots, 2d-k\}$ such that

$$j_1 < j_2 < \cdots < j_{d-k} \quad \text{and} \quad j_{d-k+1} < j_{d-k+2} < \cdots < j_{2d-2k}.$$

The sum in (22) has $\binom{2d-2k}{d-k}$ terms. The group $G = \text{SL}(n)$ acts on the vectors a and b , and hence on the span of the polynomials (22). This is the irreducible representation $S_\lambda(\mathbb{C}^n)$.

Proposition 5.2. *The polynomial (22) is in the coordinate ring of $\sigma(X)$, i.e. it lies in the image of the ring homomorphism $\mathbb{C}[x] \rightarrow \mathbb{C}[a, b]$ that is given by the parameterization (21). Its preimage in $\mathbb{C}[x]$ is unique. That polynomial vanishes on $\tau(X)$ if and only if $k \geq 4$.*

Proof. Since k is even, the polynomial (22) is invariant under permuting the letters a and b . The coefficients of the right hand side of (21) span the space of all such invariant polynomials of degree d . This follows from the fact that the usual ring of symmetric polynomials is generated by the power sums. Hence (22) is in the image of the ring map $\mathbb{C}[x] \rightarrow \mathbb{C}[a, b]$. The kernel of that map is the ideal of the secant variety $\sigma(X)$. That ideal contains no quadrics. Hence the preimage of (22) in $\mathbb{C}[x]$ is unique. The final statement follows from part (1) in the Corollary in [13, §1]. The next example illustrates that statement. \square

Example 5.3. Let $n = 2$ and $k = d$ even, so X is the rational normal curve in \mathbb{P}^d . Consider the polynomial $(a_1 b_2 - a_2 b_1)^k$. For $k = 2$, its preimage in $\mathbb{C}[x]$ is $x_0 x_2 - x_1^2$. This does not vanish on $\tau(X) = \mathbb{P}^2$. For $k = 4$, the preimage is $x_0 x_4 - 4x_1 x_3 + 3x_2^2$. This vanishes on $\tau(X)$.

For any pair (μ, ν) as in (18), we write $f_{(\mu, \nu)}$ for the unique preimage of (22) under the map $\mathbb{C}[x] \rightarrow \mathbb{C}[a, b]$. This is well-defined by Proposition 5.2. The polynomial $f_{(\mu, \nu)}$ is easily computable by solving a linear system of equations. For instance, two x -polynomials in Example 5.3 are $f_{(11, 22)}$ and $f_{(1111, 2222)}$. Or, using tableaux, we might write $f_{\begin{smallmatrix} 11 \\ 22 \end{smallmatrix}}$ and $f_{\begin{smallmatrix} 1111 \\ 2222 \end{smallmatrix}}$.

Corollary 5.4. *A basis for the quadrics that vanish on the tangential variety $\tau(X)$ of the Veronese variety X consists of the $f_{(\mu,\nu)}$ that are indexed by the SSYT of shape $\lambda = (2d-k, k)$ where $k \in \{4, 5, \dots, d\}$ is even. Their number is obtained by summing (19) over those k .*

There are no quadrics that vanish on $\tau(X)$ when $d \leq 3$. For $d \geq 4$ we have constructed an explicit basis for that space of quadrics. The dimensions of this space is given in Table 1.

| n | d | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|-----|------|-------|-------|-------|--------|--------|
| 2 | | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| 3 | | 15 | 60 | 153 | 315 | 570 | 945 | 1470 |
| 4 | | 105 | 540 | 1711 | 4270 | 9190 | 17850 | 32130 |
| 5 | | 490 | 3150 | 12145 | 36155 | 91395 | 205905 | 425425 |

Table 1: Dimension of the space of quadrics vanishing on $\tau(\nu_d(\mathbb{P}^{n-1}))$

Proposition 5.5. *Fix a Veronese variety $X = \nu_d(\mathbb{P}^{n-1})$ with $d \geq 4$. The tangential variety $\tau(X)$ is defined, as a subvariety of the secant variety $\sigma(X)$, by the quadrics in Corollary 5.4.*

Proof. This is proved in Landsberg’s book on tensors, namely in [7, Theorem 8.1.4.1]. \square

Example 5.6. For ternary quartics ($n = 3, d = 4$), we consider the 6×6 Hankel matrix

$$H = \begin{pmatrix} x_{400} & x_{220} & x_{202} & x_{310} & x_{301} & x_{211} \\ x_{220} & x_{040} & x_{022} & x_{130} & x_{121} & x_{031} \\ x_{202} & x_{022} & x_{004} & x_{112} & x_{103} & x_{013} \\ x_{310} & x_{130} & x_{112} & x_{220} & x_{211} & x_{121} \\ x_{301} & x_{121} & x_{103} & x_{211} & x_{202} & x_{112} \\ x_{211} & x_{031} & x_{013} & x_{121} & x_{112} & x_{022} \end{pmatrix}.$$

The Veronese surface $X \subset \mathbb{P}^{14}$ is defined by the 2×2 -minors of H . The 5-dimensional secant variety $\sigma(X)$ is defined by the 3×3 -minors of H . The tangential variety $\tau(X)$ is the codimension one subvariety of $\sigma(X)$ defined by the vanishing of the following 15 quadrics: $f_{(1111,2222)}$, $f_{(1111,2223)}$, $f_{(1111,2233)}$, $f_{(1111,2333)}$, $f_{(1111,3333)}$, $f_{(1112,2223)}$, $f_{(1112,2233)}$, $f_{(1112,2333)}$, $f_{(1112,3333)}$, $f_{(1122,2233)}$, $f_{(1122,2333)}$, $f_{(1122,3333)}$, $f_{(1222,2333)}$, $f_{(1222,3333)}$, $f_{(2222,3333)}$. Each of these symbols translates into a product of $k = 4$ factors as in (22), and from this we recover the quadric. For instance, $f_{(1111,2222)} = (a_1b_2 - a_2b_1)^4 = x_{400}x_{040} - 4x_{310}x_{130} + 3x_{220}^2$ and $f_{(1112,2333)} = (a_1b_2 - a_2b_1)(a_1b_3 - a_3b_1)^2(a_2b_3 - a_3b_2) = x_{310}x_{013} - x_{301}x_{022} - x_{220}x_{103} - x_{211}x_{112} + 2x_{202}x_{121}$.

Remark 5.7. The quadratic polynomials $f_{\mu,\nu}$ that cut out $\tau(X)$ do not contribute to the semi-algebraic description of the real rank two locus $\rho(X)$. Unlike the hyperdeterminants in Theorem 4.5, they do not give valid non-trivial inequalities for $\rho(X)$. For instance, in Example 5.6, the polynomial $f_{(1111,2222)}$ is non-negative on $\sigma(X)_{\mathbb{R}}$ while $f_{(1112,2333)}$ changes sign on $\rho(X)$. Here, $\rho(X)$ is defined in $\sigma(X)_{\mathbb{R}}$ by nine quartic inequalities; cf. (16) and (17).

For the remainder of this paper we set $n = 2$, so we consider symmetric $2 \times 2 \times \cdots \times 2$ -tensors. These tensors form a projective space \mathbb{P}^d , namely the space of binary forms

$$f = \sum_{i=0}^d x_i \binom{d}{i} s^{d-i} t^i. \quad (23)$$

To describe the relevant varieties, we use the following *Hankel matrix* of format $3 \times (d-1)$:

$$H = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{d-2} \\ x_1 & x_2 & x_3 & \cdots & x_{d-1} \\ x_2 & x_3 & x_4 & \cdots & x_d \end{pmatrix}. \quad (24)$$

Our three varieties of interest satisfy the inclusions $X \subset \tau(X) \subset \sigma(X)$ in \mathbb{P}^d . They are

- $X = \{\text{rank}(H) \leq 1\} =$ the rational normal curve in $\mathbb{P}^d = \{\text{binary forms } \ell^d\}$;
- $\tau(X) =$ points on tangent lines of the curve $X = \{\text{binary forms } \ell_1^{d-1} \ell_2\}$;
- $\sigma(X) = \{\text{rank}(H) \leq 2\} =$ points on secant lines of $X = \{\text{binary forms } \ell_1^d + \ell_2^d\}$.

These projective varieties have dimensions 1, 2 and 3. Their defining equations are as follows.

Corollary 5.8. *The prime ideals of X and $\sigma(X)$ are respectively generated by the 2×2 -minors and the 3×3 -minors of the Hankel matrix H in (24). The prime ideal of the tangential variety $\tau(X)$ is minimally generated by the quartic D if $d = 3$, by the cubic $\det(H)$ and the quadric $Q = x_0 x_4 - 4x_1 x_3 + 3x_2^2$ if $d = 4$, and by $\binom{d-2}{2}$ linearly independent quadrics if $d \geq 5$.*

Proof. These equations are classical and found in many sources, including Landsberg's book [7]. The ideal of $\tau(X)$ is derived from the description in Theorem 5.1 and Corollary 5.4. \square

The real rank two locus $\rho(X)$ is a 3-dimensional semi-algebraic set. It consists of binary forms $\ell_1^d + \ell_2^d$ where ℓ_1, ℓ_2 are real. Its algebraic boundary is $\tau(X)$. Theorem 4.5 implies:

Corollary 5.9. *The real rank two locus $\rho(X)$ is the subset of $\mathbb{P}_{\mathbb{R}}^d$ that is defined by the vanishing of the 3×3 -minors of H in (24) together with the following $d-2$ quartic inequalities:*

$$x_i^2 x_{i+3}^2 - 6x_i x_{i+1} x_{i+2} x_{i+3} - 3x_{i+1}^2 x_{i+2}^2 + 4x_{i+1}^3 x_{i+3} + 4x_i x_{i+2}^3 \geq 0 \quad \text{for } i = 0, 1, \dots, d-3. \quad (25)$$

Proof. We regard f as a $2 \times 2 \times \cdots \times 2$ -tensor with d factors, and we apply (16) and (17). \square

We conclude by examining the real rank two loci for binary quartics and binary quintics.

Example 5.10. Let $d = 4$. We examine the geography of the real hypersurface $\sigma(X)_{\mathbb{R}}$ in $\mathbb{P}_{\mathbb{R}}^4$. It decomposes into three semi-algebraic strata. Up to closure, these strata are: the set $\sigma(X)^{++} = \{\ell_1^4 + \ell_2^4\}$ of semi-definite real rank two quartics; the set $\sigma(X)^{+-} = \{\ell_1^4 - \ell_2^4\}$ of indefinite real rank two quartics; the set $\sigma(X)^{\text{cpx}}$ of quartics of real rank three and complex rank two. All three strata intersect in the curve $X_{\mathbb{R}}$ of rank one quartics.

We examine the points on the boundary $\partial(\rho(X))$. Such points cannot be in the closure of $\sigma(X)^{++}$. They must be in the closure of $\sigma(X)^{+-}$. A typical example is

$$s^3t = \lim_{\epsilon \rightarrow 0} \frac{1}{4\epsilon} ((s + \epsilon t)^4 - s^4) = \lim_{\epsilon \rightarrow 0} (s^3t - \epsilon^2 st^3).$$

The first limit approaches s^3t from within $\sigma(X)^{+-}$. The second limit approaches from within the real rank three locus $\sigma(X)^{\text{cpx}}$. To see this, we set $i = \sqrt{-1}$ and we note that

$$s^3t - \epsilon^2 st^3 = \frac{1}{8\epsilon i} ((s + \epsilon it)^4 - (s - \epsilon it)^4).$$

The real rank two locus $\rho(X)$ is defined by the equation $\det(H) = 0$ and two inequalities $D_0 \geq 0, D_1 \geq 0$. Here D_0 is the quartic in (17) and D_1 is obtained by replacing $x_i \mapsto x_{i+1}$ for all unknowns. The variety $V(\det(H), D_0, D_1)$ has two irreducible components, namely the line $V(x_1, x_2, x_3)$ and the surface $\tau(X) = V(\det(H), 3x_2^2 - 4x_1x_3 + x_0x_4)$. Hence, the real rank two boundary is not obtained by setting the inequalities in Corollary 5.9 to zero. Note that the rank two tensor T in (14) is symmetric and lies in $V(x_1, x_2, x_3)$.

Example 5.11. Let $d = 5$. Then $\rho(X)$ is defined by $\text{rank}(H) \leq 2$ and three inequalities $D_0, D_1, D_2 \geq 0$. The ideal of the tangential surface $\tau(X)$ is generated by three quadrics

$$Q_0 = 3x_2^2 - 4x_1x_3 + x_0x_4, \quad Q_1 = 2x_2x_3 - 3x_1x_4 + x_0x_5, \quad Q_2 = 3x_3^2 - 4x_2x_4 + x_1x_5. \quad (26)$$

It turns out that one inequality suffices to define the real rank two locus inside the rank two locus. Namely, $\rho(X)$ is the set of binary quintics given by $\text{rank}(H) \leq 2$ and $Q_1^2 - 4Q_0Q_2 \geq 0$.

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Authors’ addresses:

Anna Seigal, University of California, Berkeley, USA, seigal@berkeley.edu

Bernd Sturmfels, University of California, Berkeley, USA, bernd@berkeley.edu